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Quenched complexity of the mean-field p-spin spherical model with external magnetic field

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Abstract. We consider the p-spin spherical spin-glass model in the presence of an external magnetic field as a general example of a mean-field system where a one-step replica symmetry breaking (1-RSB) occurs. In this context we compute the complexity of the Thouless–Anderson–Palmer states, performing a quenched computation. We find what the general connection is between this method and the standard static 1-RSB one, formulating a clear mapping between the parameters used in the two different calculations. A dynamical analysis of the model confirms the validity of our results.

1. Introduction

Two different sets of results can be obtained when studying a statistical model. The first set is given by a static (thermodynamical) analysis of the system, while the second one is obtained from a dynamical approach (e.g., Langevin dynamics). In the case of disordered models as spin glasses, these two sets of results are not generally connected in a straightforward way. The reason for this is that spin glasses always exhibit an out-of-equilibrium dynamical behaviour, which, at least at the mean-field level, is believed to be related to the presence of many metastable states (whether metastability is relevant in finite-dimensional disordered systems is still an open issue [1]).

However, there is a striking correspondence between certain types of dynamical scenarios and their specific static counterparts. In other words, some particular dynamical results are always associated with the same set of static results and vice versa. For instance, in the context of the static replica symmetry breaking (RSB) scheme for mean-field spin glasses [2], we always find a dynamical asymptotic energy larger than the static one in models which are solved by a one-step solution (1-RSB), while the two energies are the same in models statically solved by a full RSB solution [3–7]. This fact suggests that there must be some kind of underlying general physical explanation for these connections.

There is a third approach that can be used in the study of disordered systems when many states are present and which has the advantage of catching some of the results of both the statics and the dynamics. This approach is purely entropic, that is it just deals with the number of states of the system, disregarding their thermodynamical Boltzmann weights. In this context, the interesting quantity is the so-called *complexity*, defined as the density of the logarithm

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of the number of states. The complexity as a function of the free energy density provides information on the structure of the phase space. It enables us to recover the usual static results, but it also takes into consideration the metastable states relevant for the dynamics [8–11].

In the context of mean-field spin glasses the complexity can be calculated in various ways. A first and intuitive method consists in simply computing the number of local minima of some mean-field free energy, a function of the local magnetizations. Another possibility is to obtain the complexity by means of an analysis which makes use of constrained systems [12,13]. Each of these methods has advantages and limits. In this paper we shall concentrate on the first one. The reason is that this method can be extended to the analysis of stationary points different from minima [14]. A deep knowledge of the whole free energy landscape, including saddles of various order, is of crucial importance to understand the dynamical behaviour of the system, since in this context it should be possible to find an explanation for the relation we stated above between statics and dynamics [15, 16].

As a first step in this direction it is necessary to understand how general and feasible this entropic approach is. Up to now it has given exact results only in a restricted set of mean-field models of spin glasses, i.e. results analytically consistent with the static and dynamical ones [10,17–19]. Yet, the physical interpretation of the parameters used has often been obscure. In particular, for what concerns systems with 1-RSB solution, the entropic approach has been used only in cases where the states are uncorrelated, so that an annealed computation has always been performed. In this kind of model there is a band of free energy densities where the number of states is exponentially large in the size of the system. It has been found that the static and the dynamic free energies (the last being defined as the free energy of the states reached by the dynamics) are equal to the lowest and highest edges of this band, respectively. We believe this to be true in any 1-RSB model. Yet, in order to find out how general these results are, it is not sufficient to consider this restricted set of models. Indeed, due to the absence of correlation among the states, these models have some particular features, which make them a special subset of the 1-RSB class. For instance, the static and dynamical transitions are both discontinuous, which is not a general property of 1-RSB models.

In this paper we extend the entropic calculation to the simplest model which displays *all* the general features of the 1-RSB class, namely the p-spin spherical model in the presence of an external magnetic field. This model is a very general paradigm of 1-RSB systems, since as a function of temperature and field it displays all the possible characteristics of these systems [3]†. From a static point of view, it has both a continuous and discontinuous transition in the overlap order parameter. From a dynamical point of view, it presents an off-equilibrium behaviour with a modified fluctuation-dissipation theorem, it has an asymptotic dynamical energy greater than the static one, and a non-vanishing asymptotic magnetization [16]. Moreover, while the p-spin with zero field is the only one-step model for which a particularly simple solution for the correlation and the response function holds, this does not happen if a magnetic field is present, the solution being, in this case, of the more complicated form common to the whole class of 1-RSB systems [5]. It is believed that 1-RSB systems are good mean-field candidates for structural glasses [16]. This is one of the reasons why these systems have recently received so much attention.

We perform the calculation of the complexity following the general method of [9]. In this context we are able to state in the most general way which is the physical interpretation of all the parameters introduced in the entropic calculation. Moreover, we find a one to one mapping between all these parameters and the ones of the 1-RSB static solution, giving a

† Another model displaying all the 1-RSB features is that of a manifold in a short-range correlated random potential [5, 20].

physical interpretation of this relation, which was until now rather unclear.

The paper is organized in the following way. In section 2 we define the model and compute the complexity. In the next section we discuss the results and find the relation with the static 1-RSB calculation. In section 4 we study the behaviour of the complexity as a function of temperature and magnetic field. In section 5 we perform a dynamical analysis of the model and compare it with the results coming from the complexity. Finally, we state our conclusions in section 6.

2. Model and complexity

The p-spin spherical model is defined by the Hamiltonian,

$$H(s) = -\sum_{i_1<\cdots<i_p} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} - h\sum_i s_i. \tag{1}$$

The spins s are real variables satisfying the spherical constraint $\sum_i s_i^2 = N$, where N is the size of the system. The couplings J are Gaussian variables with zero mean and variance $p!/2N^{p-1}$ and h is an external magnetic field [3, 21, 22]. In the context of the Thouless–Anderson–Palmer (TAP) approach [8], it is possible to formulate a set of mean-field equations for the local magnetizations $m_i = \langle s_i\rangle$. A mean-field free energy density $f_{\rm TAP}$, function of the magnetizations m_i, was introduced for this model in [11],

$$f_{\rm TAP} = -\frac{1}{N}\sum_{i_1<\cdots<i_p} J_{i_1\ldots i_p} m_{i_1}\ldots m_{i_p} - \frac{h}{N}\sum_i m_i - \frac{1}{2\beta}\log(1-q) + g(q) \tag{2}$$

where $q = N^{-1}\sum_i m_i^2$ is the self-overlap related to the magnetization m and $g(q) = -\frac{\beta}{4}[(p-1)q^p - pq^{p-1} + 1]$ is the Onsager reaction term (for a derivation of the TAP free energy see also [10, 17]). The minimization of (2) with respect to the $\{m_i\}$ gives the TAP equations of the system,

$$\mathcal{T}_k(m) \equiv -p\sum_{i_2<\cdots<i_p} J_{k,i_2\ldots i_p} m_{i_2}\ldots m_{i_p} - h + 2m_k\left(\frac{1}{2\beta(1-q)} + g'(q)\right) = 0. \tag{3}$$

In the low-temperature phase these equations admit many possible solutions, corresponding to different stationary points of the mean-field free energy (2). The minima among all the stationary points can be identified with stable and metastable states of the system.

The *complexity* $\Sigma(f)$ is defined in the following way,

$$\Sigma(f) \equiv \lim_{N\to\infty}\frac{1}{N}\overline{\log\mathcal{N}(f)} \tag{4}$$

where $\mathcal{N}(f)$ is the number of local *minima* of the free energy (2), that is, the number of states of the system with free energy density f. We average the logarithm of $\mathcal{N}$ since we expect this to be the extensive quantity. In order to perform this average it is necessary to introduce replicas. However, when the external field h is set equal to zero, the correct ansatz for the overlap matrix turns out to be symmetric and diagonal. This is equivalent to directly averaging the number $\mathcal{N}$ of solutions (annealed average) [10]. The physical reason for this is that when $h = 0$ there is no preferred direction in the phase space sphere, thus the typical states are orthogonal to each other and their mutual overlap is zero. On the other hand, when $h \neq 0$ there is a migration of the states towards the direction of the magnetic field and their mutual overlap is different from zero. In this case the quenched average has to be performed. Anyhow, as it is shown below, due to the particular nature of the calculation, the correct ansatz for the overlap matrix is simpler than the 1-RSB used in the statics [3].

In order to compute Σ we use the replica trick,

$$\overline{\log \mathcal{N}} = \lim_{n\to 0} \frac{1}{n} \log \overline{\prod_{a=1}^{n} \mathcal{N}^a} \tag{5}$$

where each $\mathcal{N}^a$ is given by

$$\mathcal{N}^a(f) = \int \mathcal{D}m^a \prod_{k=1}^{N} \delta(\mathcal{T}_k(m^a)) |\det \mathcal{A}(m^a)| \delta(f_{\mathrm{TAP}}(m^a) - f) \tag{6}$$

and $\mathcal{A}_{kl}(m) = \partial_k \partial_l f_{\mathrm{TAP}}(m)$ is the Hessian of the TAP free energy evaluated in a particular solution m. In what follows the symbol $\mathcal{D}x$ refers to the integration over all the site variables, $\mathcal{D}x = \mathrm{d}x_1 \ldots \mathrm{d}x_N$. Let us introduce the following Grassmann representation for the determinant,

$$\det \mathcal{A}(m^a) = \int \mathcal{D}\bar{\psi}^a \mathcal{D}\psi^a \exp[-\bar{\psi}^a \mathcal{A}(m^a)\psi^a] \tag{7}$$

and a bosonic representation for the delta functions,

$$\prod_{k=1}^{N} \delta(\mathcal{T}_k(m^a)) = \int \mathcal{D}\lambda^a \exp(-\lambda^a \mathcal{T}(m^a)) \tag{8}$$

$$\delta(f_{\mathrm{TAP}}(m^a) - f) = \int \mathrm{d}\omega^a \exp[-\omega^a (f_{\mathrm{TAP}}(m^a) - f)] \tag{9}$$

where the integrals over the variables λ and ω are on the imaginary axis. The sums over site indices are always understood whenever site dependent quantities are mutually multiplied. In the rest of the calculation we shall disregard the modulus of the determinant in (6). This approximation is safe for zero magnetic field, as long as we are counting stationary points with a given *fixed* free energy density [14]. Indeed, it can be shown that in this way only the minima of f_{TAP} are actually taken into consideration [14]. We assume this to also hold with $h \neq 0$. We then have,

$$\Sigma(f) = \lim_{n\to 0} \frac{1}{nN} \log \int \mathcal{D}m \mathcal{D}\lambda \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \mathrm{d}\omega \, \overline{\exp[-\mathcal{S}_J(m, \lambda, \bar{\psi}, \psi, \omega)]} \tag{10}$$

where the action $\mathcal{S}_J$ is given by

$$\mathcal{S}_J = \sum_{a=1}^{n} [\lambda^a \mathcal{T}(m^a) + \bar{\psi}^a \mathcal{A}(m^a)\psi^a + N\omega^a (f_{\mathrm{TAP}}(m^a) - f)]. \tag{11}$$

The average over the disorder generates couplings between variables with different replica indices. It is convenient to express these terms by means of the following overlap matrices, which can be introduced in the usual way [3, 10, 14],

$$NS_{ab} = m^a m^b \qquad NL_{ab} = \lambda^a \lambda^b \qquad NR_{ab} = m^a \lambda^b \qquad NT_{ab} = -\bar{\psi}^a \psi^b. \tag{12}$$

In terms of the overlap matrices (12) we have,

$$\Sigma(f) = \lim_{n\to 0} \frac{1}{nN} \log \int \mathcal{D}S \mathcal{D}L \mathcal{D}R \mathcal{D}T \, \mathrm{d}\omega \exp[-N\mathcal{S}(S, L, R, T, \omega)] \tag{13}$$

where the effective action is given by

$$-\mathcal{S} = \frac{p}{4} \sum_{ab} L_{ab} S_{ab}^{p-1} + \frac{p(p-1)}{4} \sum_{ab} (R_{ab}^2 - T_{ab}^2) S_{ab}^{p-2} + \frac{h^2}{2} \sum_{ab} L_{ab}$$
$$+ \sum_a 2 \left[g'(S_{aa}) + \frac{1}{2\beta(1 - S_{aa})} \right] (T_{aa} - R_{aa}) - \log\det T + \frac{1}{2} \log\det S$$

$$+\frac{1}{2}\log\det(R^T S^{-1} R - L) + \frac{p}{2}\omega\sum_{ab} R_{ab} S_{ab}^{p-1} + h^2\omega\sum_{ab} R_{ab}$$
$$+nf\omega - \omega\sum_a \left[g(S_{aa}) - \frac{1}{2\beta}\log(1 - S_{aa}) \right] + \frac{1}{4}\omega^2\sum_{ab} S_{ab}^p + \frac{h^2}{2}\omega^2\sum_{ab} S_{ab}. \tag{14}$$

We have assumed $\omega_a = \omega$ since it depends only on one replica index. As usual, we compute the integral (13) by means of a saddle point approximation, so that,

$$\Sigma(f) = \lim_{n\to 0}\frac{1}{n}\mathrm{Ext}_\phi[-\mathcal{S}(f,\phi)] \tag{15}$$

where ϕ stands for all the variational parameters, S, L, R, T and ω.

In order to simplify the saddle point equations for the overlap matrices we make use of the fact that action (11) is invariant under the following Becchi–Rouet–Stora–Tyutin (BRST) transformation [23–25],

$$\begin{aligned} &m_i^a \to m_i^a + \epsilon\psi_i^a \qquad \bar\psi_i^a \to \bar\psi_i^a - \epsilon\lambda_i^a \qquad \lambda_i^a \to \lambda_i^a - \omega^a\epsilon\psi_i^a \\ &\psi_i^a \to \psi_i^a \qquad \omega^a \to \omega^a \end{aligned} \tag{16}$$

where ϵ is a constant Grassmann parameter. As a consequence, for each operator O, which is a function of the variables m^a, λ^a, $\bar\psi^a$, ψ^a and ω^a, we have that $\langle \delta O\rangle = 0$, where δO is the variation of O under (16) and the brackets indicate an average over the measure defined by the action $\mathcal{S}_J$. If we consider the two cases $O = m^b\bar\psi^a$ and $O = \lambda^b\bar\psi^a$, we immediately get the equations $\langle\bar\psi^a\psi^b\rangle = -\langle m^b\lambda^a\rangle$ and $\langle\omega\bar\psi^a\psi^b\rangle = \langle\lambda^a\lambda^b\rangle$. From the definitions (12) we then obtain the following relations for the saddle point values of the overlap matrices,

$$R_{ab} = T_{ab} \qquad L_{ab} = -\omega T_{ab} \tag{17}$$

which simplify a lot of the calculation. At this point we have to choose an ansatz for the overlap matrices in order to solve the saddle point equations. Let us consider first the matrix S, which has an explicit physical meaning. From the definition (12) we see that the diagonal element S_{aa} corresponds to the self-overlap of an individual TAP solution. On the other hand the off-diagonal elements S_{ab} correspond to mutual overlaps between different TAP solutions. Given this we assume for the matrix S the simplest form consistent with this interpretation, that is a symmetric matrix,

$$S_{ab} = (s_1 - s_0)\delta_{ab} + s_0. \tag{18}$$

In the saddle point, s_1 and s_0 represent respectively the self-overlap and the mutual overlap of the states with free energy density f. As discussed above, in absence of the magnetic field s_0 vanishes, the matrix S is diagonal and the quenched average coincides with the annealed one. On the other hand in the presence of the field we expect a value $s_0 \neq 0$. Consistently with the ansatz used for S we set,

$$T_{ab} = (t_1 - t_0)\delta_{ab} + t_0. \tag{19}$$

Using the BRST relations (17) and the above form for the matrices S and T, we can reduce the saddle point equations to the following five coupled equations,

$$\begin{aligned}
0 &= -\omega\left(\frac{p}{2}s_1^{p-1}+h^2\right)-\frac{p(p-1)}{2}t_1 s_1^{p-2}-\frac{1}{z}+\frac{1}{z^2}(\omega s_0+t_0)+2g'(s_1)+\frac{1}{\beta(1-s_1)}\\
0 &= \omega\left(\frac{p}{2}s_0^{p-1}+h^2\right)+\frac{p(p-1)}{2}t_0 s_0^{p-2}-\frac{1}{z^2}(\omega s_0+t_0)\\
0 &= -\omega\left(\frac{p}{2}s_1^{p-1}+h^2\right)+\frac{1}{t_1-t_0}-\frac{t_0}{(t_1-t_0)^2}-\frac{1}{z}+\frac{1}{z^2}(\omega s_0+t_0)\\
0 &= \omega\left(\frac{p}{2}s_0^{p-1}+h^2\right)+\frac{t_0}{(t_1-t_0)^2}-\frac{1}{z^2}(\omega s_0+t_0)\\
0 &= \frac{p}{4}(t_1 s_1^{p-1}-t_0 s_0^{p-1})+\omega\left(\frac{1}{2}(s_1^{p-1}-s_0^{p-1})+h^2(s_1-s_0)\right)\\
&\quad -\frac{s_1}{2z}+\frac{\omega s_0+t_0}{2z^2}(s_1-s_0)+f-g(s_1)+\frac{1}{2\beta}\log(1-s_1)
\end{aligned} \tag{20}$$

where $z=[\omega(s_1-s_0)+t_1-t_0]$ and $g(s)$ is the Onsager term of (2). It is easy to verify that for $h=0$ these equations give the standard result of [10].

3. Connection with the statics

We are now in the position to compute the complexity Σ in every point of the plane (T,h) and, in particular, in the regime where the statics of the model displays a 1-RSB solution (low temperatures and fields) [3]. As we will show below, our results clarify the relation between the present entropic approach and the usual static one.

First we note that equations (20) give, after some algebra, the relations,

$$t_0=0 \qquad t_1=\beta(1-s_1). \tag{21}$$

The physical interpretation of these equations is very simple. The saddle point value of the matrix T_{ab} is related to the expectation value $\langle\bar{\psi}^a\psi^b\rangle$ (see equation (12)) and from (7) it is clear that this expectation value is nothing more than the average of the matrix $\mathcal{A}^{-1}$, which is by definition the inverse of the Hessian of the TAP free energy. Therefore, the parameter t_1 can be interpreted as the inverse curvature of the free energy in a typical state with free energy f. On the other hand, since s_1 is the self-overlap of this state, $\beta(1-s_1)$ is the the magnetic susceptibility χ. In this way equation (21) gives the expected static relation between the fluctuation t_1 and the dissipation χ of an equilibrium system, consistently with the calculations of [9, 19].

Due to equation (21) we are left with only three parameters, s_1, s_0 and ω. As we have seen, the physical interpretation of s_1 and s_0 is straightforward, while this is still not the case for ω. In order to better understand what the role of ω is we note that in the saddle point we have (see equations (15) and (14)),

$$\frac{\mathrm{d}\Sigma(f)}{\mathrm{d}f}=\omega(f). \tag{22}$$

If we call f_0 the ground state free energy, defined by $\Sigma(f_0)=0$, the number of states having extensive free energy $Nf=Nf_0+\delta f$, is

$$\mathcal{N}(\delta f)\sim \mathrm{e}^{\Sigma'(f_0)\,\delta f}=\mathrm{e}^{\omega(f_0)\,\delta f} \qquad \delta f\sim \mathrm{O}(1). \tag{23}$$

This equation shows that $\omega(f_0)$ plays the same role as the static parameter x, which represents the breaking point in the 1-RSB scheme [10, 26, 27]. Given this, it is clear that, for each value

of (T, h), the 1-RSB computation and the present one must map one into the other with the following rule:

$$f_0 = f_{\rm RSB} \Rightarrow s_1(f_0) = q_1 \qquad s_0(f_0) = q_0 \qquad \omega(f_0) = \beta x \tag{24}$$

where $f_{\rm RSB}$ is the static 1-RSB equilibrium free energy and q_1, q_0 and x are the 1-RSB parameters of the static computation (respectively, self-overlap, overlap and breaking point) [3]. It is possible to verify analytically that the relations (24) are fulfilled by our saddle point equations: the complexity is zero at the static free energy $f_{\rm RSB}$ and here the equations for the parameters s_1, s_0 and ω reduce exactly to the static equations for q_1, q_0 and βx.

In this way we have shown that, despite the symmetric form of the overlap matrices used in our calculation (see equations (18) and (19)), the complexity describes the statics of the model with the same degree of accuracy as the 1-RSB calculation. This point has always been rather mysterious, since it was not evident why the two approaches should give the same results. Now the explanation is clear. At the ground state free energy density f_0 the two calculations simply write the same quantities in different ways. In the RSB approach the self-overlap q_1 of the states is introduced, while for the complexity it is natural to deal with the curvature of the states, because of the presence of the Hessian. Yet, these two quantities are, as we have seen, trivially related. Moreover, the breaking point x, which is typical of the RSB calculations, has its counterpart in the parameter ω, which has a very simple interpretation in the context of the complexity, being just the derivative of Σ with respect to the free energy. Finally, the overlap matrix in the two cases has a slightly different physical meaning and this is why different ansatz are taken for them: in the RSB approach this matrix refers to the overlap between *configurations*, while in the present context it refers to the overlap between *states*. Due to this, the present approach has *one less step* of replica symmetry breaking, compared with the standard static one. Clearly, the price we have to pay for having a symmetric overlap matrix is the knowledge of the explicit form of the TAP free energy. Interestingly enough, the parameter x drops from the symmetric overlap matrix, to reappear in disguise as ω. We believe that all the relations we found are valid for all 1-RSB models. The reason why they have not been recognized in the past is twofold. In the simple case of $h = 0$, due to the homogeneity of the model, the complexity does not depend on the temperature, so that the calculation can be carried out at $T = 0$ [10], where $s_1 = 1$ and equation (21) is singular. Besides, when the simpler annealed computation was performed many parameters were integrated out for convenience, so that relations (21) and (24) were not detected.

We stress that what is stated above only holds for $f = f_0$, where the complexity is zero. On the other hand, increasing f provides information on the whole spectrum of the states.

For the sake of completeness, we would like to recall that a different method exists by which the complexity can be computed [12, 13]. This method relates the complexity $\Sigma(f)$ to the static free energy of r real replicas of the system in the following way: $\Sigma(r) = \beta r^2 \frac{\partial F_r}{\partial r}$ and $f(r) = \frac{\partial [r F_r]}{\partial r}$, where $r F_r$ is the free energy density of the r coupled real replicas in the limit where the coupling goes to zero. To obtain Σ it is therefore necessary to compute F_r. For systems with 1-RSB one usually assumes a one step of breaking also for the overlap matrix related to F_r, with a fixed breaking parameter equal to r and two variational parameters q_1 and q_0 [12, 19]. In this case the expression of F_r as a function of q_1, q_0 and r is analogous to the usual static one, except for the fact that it has to be minimized with respect to q_0 and q_1, but not with respect to r. With these assumptions it is possible to obtain an explicit expression of Σ in terms of f, q_1, q_0 and r, where q_1 and q_0 are given by their saddle point values, while r is fixed by the above equation for $f(r)$. If we now compare this expression for Σ with the one coming from our calculation, we find that they are perfectly consistent: the equations for q_1, q_0 and r are exactly the same as the ones for s_1, s_0 and $\beta\omega$, and $\Sigma(r)$ coincides with (15).

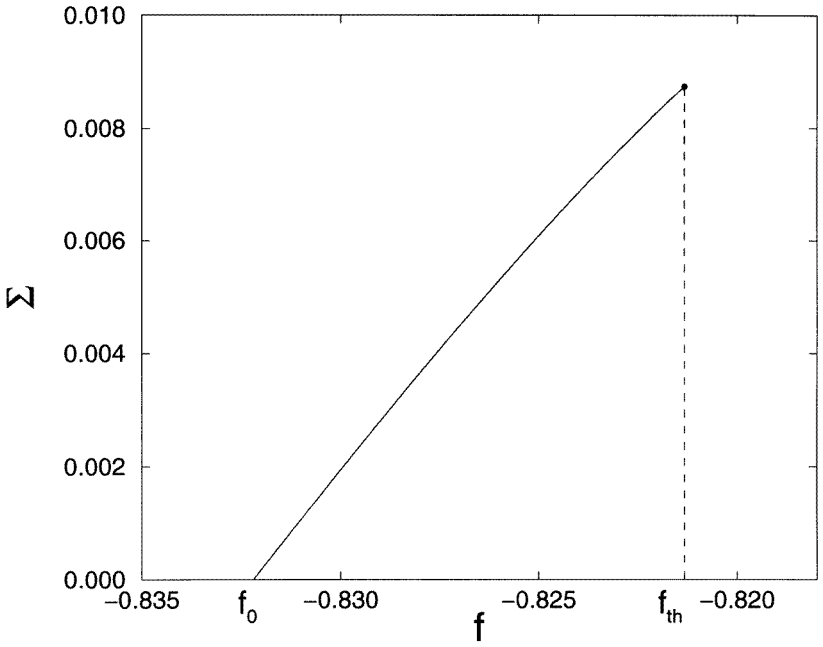

Figure 1. The complexity $\Sigma(f)$ as a function of the free energy density f for $T = 0.2$, $h = 0.2$ and $p = 3$. The minimum value $\Sigma = 0$ occurs at $f_0 = f_{\mathrm{RSB}} = -0.8322$. The maximum value is at $f_{\mathrm{th}} = -0.8213$, which is the free energy density of the states reached by the dynamics.

Finally, we note that our results are consistent with the analysis recently performed in [28], where a thermodynamical description of the glassy transition for this model is presented.

4. Behaviour of the complexity in the temperature and field plane

From equations (14), (15) and (21) we obtain the explicit expression of the complexity

$$\Sigma(f) = \frac{p}{4}\omega t_1 s_1^{p-1} + \frac{h^2}{2}\omega t_1 - \frac{1}{2}\log t_1 + \frac{1}{2}\log z + \frac{1}{2}\frac{\omega s_0}{z} + \omega\left(f - g(s_1) + \frac{1}{2\beta}\log(1-s_1)\right) + \frac{\omega^2}{4}(s_1^p - s_0^p) + \frac{h^2}{2}\omega^2(s_1 - s_0) \tag{25}$$

with $t_1 = \beta(1 - s_1)$ and s_1, s_0 and ω given by the saddle point equations (20).

Let us now analyse the behaviour of the complexity in the (T, h) plane. For small enough values of temperature and field many pure states are present, therefore we expect the complexity to be different from zero in a finite range of free energy densities. Actually, this is what happens. In figure 1 we show $\Sigma(f)$ as a function of f at fixed small values of T and h: Σ is defined between a minimum free energy density f_0 where it is zero, and a threshold free energy density f_{th} where it takes its maximum value. Since for high values of temperature and field only one (finite magnetization) paramagnetic state is present [3], we expect that by increasing T and h the number of existing states progressively decreases until one single state is left. Indeed, if we look at $\Sigma(f)$ at a fixed h, but at different values of T, we find that the interval of free energies $[f_0, f_{\mathrm{th}}]$ where Σ is defined, becomes smaller and smaller as the temperature is increased, and finally it shrinks to a single point, with $\Sigma = 0$, at a certain critical temperature $T_c(h)$. For $T > T_c(h)$, $\Sigma(f)$ is defined only in one point, where $\Sigma = 0$, and f here coincides with the

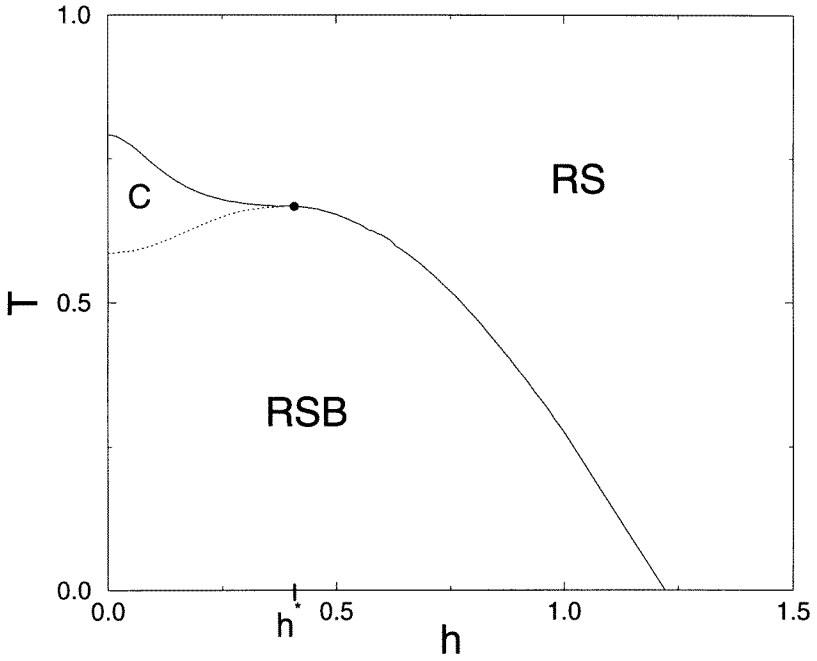

Figure 2. The full curve corresponds to the critical temperature line $T_c(h)$ where Σ reduces to a point and the RSB static solution disappears, for $p = 3$. The dotted curve corresponds to the static transition line. The two curves are coincident for fields $h > h^\star = 0.408$. The static transition is continuous for $h > h^\star$, and discontinuous for $h < h^\star$. The region C is the region of coexistence of the RS and RSB solutions.

replica symmetric static free energy, that is, the free energy of the single paramagnetic state. In figure 2 the line $T_c(h)$ is shown in the (T, h) plane†. This line separates the region of the plane where only one pure state exists ($T > T_c(h)$) from the region where many pure states are present ($T < T_c(h)$) and therefore we identify it as the *geometrical* transition line. We note that this line is monotonically decreasing with the field, so that there is no 're-entrance' [29]. We find that the geometrical transition line coincides with the line where the RSB static solution ceases to exist [3]. This indicates that, as in the case with $h = 0$ [11], the states with the lowest free energy density are the last to disappear. We note that the critical line $T_c(h)$ does not coincide in the whole plane with the transition line given by the statics (dotted curve in figure 2) [3]. Indeed the two lines only coincide for $h > h^\star$, where the static transition is continuous ($q_1 = q_0$ at the transition), while they are strictly different for $h < h^\star$ (discontinuous transition) [3]. This is consistent with the different physical scenarios corresponding to the continuous and the discontinuous transitions. In the first case, the transition corresponds to different states which merge into a unique ergodic component, so above the transition line only one state is present. In the second case, many pure states with finite complexity exist both above and below the transition line, and the transition corresponds to the point where the lowest states ($f = f_0 = f_{\mathrm{RSB}}$) become the relevant ones from a thermodynamical point of view. In this regime, just above the static transition line equilibrium is given by a class of states with finite complexity. In the replica scheme the right solution is the replica symmetric one, which corresponds to a bunch of exponentially numerous different pure states [10, 11].

† We thank David Sherrington for first suggesting the existence of such a line in the phase $h < h^\star$.

Above the geometrical transition line all the states but the paramagnetic one disappear and the RS solution corresponds in this case to a single ergodic component.

5. Threshold energy and dynamical behaviour

As mentioned in the previous section, at fixed temperature and field, the complexity is defined within a certain range $[f_0, f_{\rm th}]$ of free energy densities. It is well known that at $h = 0$ in the low-temperature region the states with $f = f_{\rm th}$ are the dynamically relevant ones, that is, their energy density coincides with the asymptotic value of the dynamical energy [4, 11, 22, 30–32]. In this section we show that this also holds at non-zero values of the field, as expected. In order to check this point we perform a dynamical analysis of the model with $h \neq 0$ and compare the dynamical asymptotic energy with the energy of the threshold states of the complexity. The equivalence between the threshold and dynamical energies can be confirmed numerically for any temperature and field, but for the sake of simplicity we will only quote the analytical results at zero temperature and small field. If we take the limit $T \to 0$ of equations (20) and expand it for small fields we get the following expression for the energy density of the threshold states,

$$E_{\rm th} = -\sqrt{\frac{2(p-1)}{p}}\left(1 + \frac{(p-1)}{p}h^2\right) + \mathrm{O}(h^3) \qquad T \to 0. \tag{26}$$

Let us now compute the asymptotic dynamical energy [4] (for a more complete dynamical analysis in similar cases see [5, 29]). The relaxational dynamics for the system is given by the Langevin equations,

$$\dot{s}_i(t) = -\beta\frac{\delta H}{\delta s_i(t)} - y(t)s_i(t) + \nu_i(t) \tag{27}$$

where $y(t)$ is a Lagrange multiplier which enforces the spherical constraint, and $\nu_i(t)$ is a Gaussian noise with zero mean and variance 2. The dynamics is completely determined by the equations for the two-time correlation and response functions, $C(t,t') = \frac{1}{N}\sum_i \overline{\langle s_i(t)s_i(t')\rangle}$ and $G(t,t') = \frac{1}{N}\sum_i \overline{\frac{\partial\langle s_i(t)\rangle}{\partial h_i(t')}}$, and for the magnetization $m(t) = \frac{1}{N}\sum_i \overline{\langle s_i(t)\rangle}$. These equations are [4, 33],

$$\partial_t C(t,t') = -y(t)C(t,t') + \mu\int_0^{t'} \mathrm{d}t''\, C^{p-1}(t,t'')G(t',t'') + \mu(p-1)\int_0^{t} \mathrm{d}t''\, G(t,t'')C^{p-2}(t,t'')C(t'',t') + \beta h m(t') \tag{28}$$

$$\partial_t G(t,t') = -y(t)G(t,t') + \delta(t-t') + \mu(p-1)\int_{t'}^{t} \mathrm{d}t''\, G(t,t'')C^{p-2}(t,t'')G(t'',t') \tag{29}$$

$$\dot{m}(t) = -y(t)m(t) + \mu(p-1)\int_0^t \mathrm{d}t''\, G(t,t'')C^{p-2}(t,t'')m(t'') + \beta h \tag{30}$$

where $\mu = p\beta^2/2$. Another equation for $y(t)$ is obtained by self-consistently exploiting the spherical constraint,

$$y(t) = 1 - p\beta\mathcal{E}(t) - (p-1)\beta h m(t) \tag{31}$$

where $\mathcal{E}(t) = N^{-1}\overline{\langle H(t)\rangle}$ is the dynamical energy density. In the limit $t \to \infty$ we have the following expression for the asymptotic dynamical energy $\mathcal{E}_\infty = \lim_{t\to\infty}\mathcal{E}(t)$ (see also [32]),

$$\mathcal{E}_\infty = \frac{1 - y_\infty}{p\beta} - \frac{p-1}{p}hm_\infty \tag{32}$$

where y_∞ and m_∞ are the corresponding asymptotic values for the Lagrange multiplier and the magnetization. In order to compute $\mathcal{E}_\infty$ we then have to find y_∞ and m_∞ by solving asymptotically the set of dynamical equations (28)–(30).

In [4] it has been shown that for large times there are two regimes for the correlation and response functions. The first regime corresponds to time separations $\tau = (t - t')$ such that $\tau/t \to 0$, where the equilibrium fluctuation dissipation theorem (FDT) holds, so that $G_{\rm FDT}(\tau) = -\partial_\tau C_{\rm FDT}(\tau)$. The second regime, known as the aging regime, corresponds to $\tau/t \sim O(1)$. Here time translation invariance is violated and FDT cannot be applied. The response function is related to the correlation function by $G_{\rm ag}(t, t') = x_d \partial_{t'} C_{\rm ag}(t, t')$, where x_d parametrizes the violation of FDT. The asymptotic values for the correlation functions in both regimes are,

$$\lim_{\tau\to\infty} C_{\rm FDT}(\tau) = \lim_{t'/t\to 1} C_{\rm ag}(t, t') = q \qquad \lim_{t'/t\to 0} C_{\rm ag}(t, t') = q_0. \tag{33}$$

In the limit $\tau \to \infty$, the equation for $C_{\rm FDT}(\tau)$ yields a relation between y_∞ and q which reads,

$$y_\infty = (1 - q)^{-1} + \mu(1 - q^{p-1}). \tag{34}$$

If we take the limit $t'/t \to 1$ in the equation for $G_{\rm ag}(t, t')$ we obtain the equation for q,

$$\mu q^{p-2}(1 - q)^2 - (p - 1)^{-1} = 0. \tag{35}$$

It is now clear that neither q nor y_∞ depend on the field. Therefore, all the dependence of the energy on the field is given by the second term of equation (32), while the first term corresponds to the asymptotic energy in zero field [4].

To obtain the asymptotic magnetization m_∞ we need to solve the coupled equations for m_∞, q_0 and x_d coming from equation (30), and from the limits $t'/t \to 0$ and $t'/t \to 1$ of the equation for $C_{\rm ag}(t, t')$,

$$0 = -y_\infty m_\infty + \mu[1 - q^{p-1}(1 - x)]m_\infty + \beta h \tag{36}$$

$$0 = -y_\infty q_0 + \mu(1 - q)q_0^{p-1} + \mu q_0(1 - q^{p-1}) + \mu x_d(q^{p-1} - q_0^{p-1}) + \beta h m_\infty \tag{37}$$

$$0 = -y_\infty + 1 + \mu(x - 1)q^p + \mu(1 - x q_0^p) + \beta h m_\infty. \tag{38}$$

For small fields we are able to find the solution analytically, which has the following simple form,

$$m_\infty = (p - 1)(1 - q)\beta h + \mathrm{O}(h^2) \tag{39}$$

$$q_0 = \mathrm{O}(h^2) \tag{40}$$

$$x_d = \frac{(p - 2)(1 - q)}{q} + \mathrm{O}(h^2). \tag{41}$$

From here we obtain the dynamical energy,

$$\mathcal{E}_\infty = -\frac{\beta}{2}\left[1 - q^p\left(1 - \frac{(p - 2)(1 - q)}{q}\right)\right] - \frac{(p - 1)^2(1 - q)}{p}h^2 + \mathrm{O}(h^3). \tag{42}$$

In the limit $T \to 0$ it can be seen from equations (35), (42) and (26) that $\mathcal{E}_\infty = E_{\rm th}$. As already said, it is possible to check numerically that these two energies are the same in the whole plane (T, h).

6. Conclusions

In this paper we have computed the complexity Σ of the states in a very general 1-RSB mean-field model for disordered systems, namely, the p-spin spherical model in presence of an external magnetic field. We stress that the introduction of the magnetic field changes the

features of this model dramatically. First, the homogeneity of the zero-field case is lost, and together with it also the non-chaoticity of the TAP solutions and consequently the possibility to exclude the temperature from the computation of the complexity [10,11]. Secondly, by varying field and temperature the model exhibits both a discontinuous and continuous transition, while only the first kind is found if $h = 0$. Furthermore, as previously pointed out, the field introduces a correlation among states, which forces us to perform a quenched computation. For this reason we had to adopt a particular ansatz and find its physical grounding, which made the connections with the static approach clearer. Moreover, the calculation becomes much more tricky from a technical point of view. Similarly, while the dynamics in zero field turns out to be a very special case in the class of 1RSB models, on the other hand when $h \neq 0$ one obtains much more general and difficult dynamical equations [5].

In our analysis we find a region of the (T, h) plane where Σ is a monotonic increasing function of the free energy density f, defined in an interval $[f_0, f_{\rm th}]$. The lower band edge f_0, defined by $\Sigma(f_0) = 0$, is always equal to the free energy density of the static 1-RSB solution, that is, $f_0 = f_{\rm RSB}$. On the other hand, the threshold value $f_{\rm th}$ gives the free energy density of the states reached asymptotically by the non-equilibrium dynamical evolution of the system. Moreover, we find a one-to-one mapping between the equations and the parameters of the complexity at $f = f_0$ and the ones of the 1-RSB static approach. This shows that the two calculations are essentially the same, even if the ansatz taken for the overlap matrices seem so different in the two cases.

We would like to add some further comments on what the main interest of this work is. There are different methods by which the complexity of a model can be computed [9, 12, 13]. The one used in this paper is simple and intuitive, since it only counts the number of minima of the mean-field free energy function. However, from a technical point of view, this method is far from being straightforward and up to now a clear quenched calculation of the complexity was lacking. Many efforts have been made in this direction for the Sherrington–Kirkpatrick model [6], which has a full RSB static solution [19, 34–36]. However, in that case neither the physical meaning of the adopted ansatz nor the eventual consistency of the lower band edge with the static free energy density were completely satisfying. In this paper, in the context of a simpler 1-RSB model, we find in a clear way how the quenched computation has to be performed and what the meaning of the used ansatz is. Therefore, we now believe we have this method under control.

It would be interesting to use this same method for a deeper analysis of the free energy landscape. Indeed, it is generally believed that in the dynamics of glassy systems a crucial role is played not only by the minima of the Hamiltonian, but also by unstable stationary points [14, 15]. In this context, an entropic computation of stationary points of any nature is of primary interest. To our knowledge, the only method suitable for such an investigation is the direct one exposed in this paper.

An entropic analysis of the states is evidently fruitful for models which present a great number of different pure states, as mean-field models of spin glasses. Whether this is the case for short range spin glasses is still an open debate†. In any case, looking at the stationary points of the Hamiltonian to better understand the dynamical behaviour still remains, at least in principle, a reasonable issue for finite-dimensional systems.

We believe this to be a strong motivation for a better understanding of the present formalism, as the one we have reached in this paper.

† For recent discussions on this topic, see for example, [37].

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